Complementary shear and transversal elongation modes in Generalized Beam Theory (GBT) for thin-walled circular cross-sections

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ABSTRACT: This paper presents the theoretical fundaments, as well as the practical procedure to evaluate the complementary modes of Generalized Beam Theory (GBT) concerning the shear and transversal elongation energies of membrane's behaviour in pipes and thin-walled hollow circular cross-sections. This study outgrows the classical assumptions of GBT, which impose no participation of shear and transversal elongation to membrane's strain energy. Although recent developments already address this problem for both open and close segmented cross-sections, the solution for the case of thin-walled circular cross-section was still unsolved. As results, these complementary modes reach the following highlights: i) the complementary membrane shear deformations in GBT not only lead to Timoshenko Beam Theory for the lower bending modes, but also can express high order shear deformation in the respective GBT's high modes; ii) the transversal elongation and the shear energies of membrane's behaviour are coupled in each mode and also coupled with the other strain energies, such as plate's bending and membrane's longitudinal elongation, only by Poisson's effect; iii) the introduction of these complementary modes in non-linear GBT analysis leads to the initial shear and transversal stress matrices. To illustrate the present procedure, an example applied in a gas pipeline system is carried out and its final results are compared with a full shell element model.

1 INTRODUCTION

Generalized Beam Theory (GBT) is a structural mechanics theory that involves 2D features of shell theory into 1D beam theory. Initially, its creator, Richard Schardt (Schardt 1989), developed it as an extension of Vlasov beam theory (Vlasov 1961). Moreover, Schardt showed that GBT can be not only extended to non-linear analysis (Schardt 1994a, Schardt 1994b, Davies 1994), but also it can include the analysis of hollow circular cross-sections (Schardt 1989, Silvestre 2007).

Mechanic effects usually only achieved by shell element analysis, such as warping and ovalization (Silvestre 2003, Silvestre 2012, Miranda & Ubertini 2013), are described in GBT by simple superposition of orthogonal modes, which are based on Fourier-Series for hollow circular cross-section. Furthermore, following the concept of separation of variables, GBT uses amplification functions in dependent longitudinal beam direction. As a result, this theory presents the numerical performance of beam element analysis together with the result's quality of shell element analysis.

Besides these attractive properties, traditional GBT has its limitations. Among them, one can highlight the neglect of shear and transversal elongation energies of membrane's behavior. Although Gonçalves (Bebiano 2014) already developed the solution for segmented cross-sections, thin-walled circular cross-section still needs this development, which has the proposed approach presented here.

2 GBT ANALYSIS OF HOLLOW CIRCULAR CROSS-SECTION

2.1 GBT's assumption for the displacement field

The displacement fields in GBT has two major ideas: i) the superposition of orthogonal modal cross-section displacement/deformation functions for the longitudinal $u(\theta)$, tangential $v(\theta)$ and transversal perpendicular $w(\theta)$ directions (as illustrated in figure 1.a); ii) an amplification function $V(x)$ of these displacements functions along the beam length. Thus, the displacement field in traditional GBT analysis (without membrane's shear and transversal elongation deformation energies) in the middle-line of hollow circular cross-section

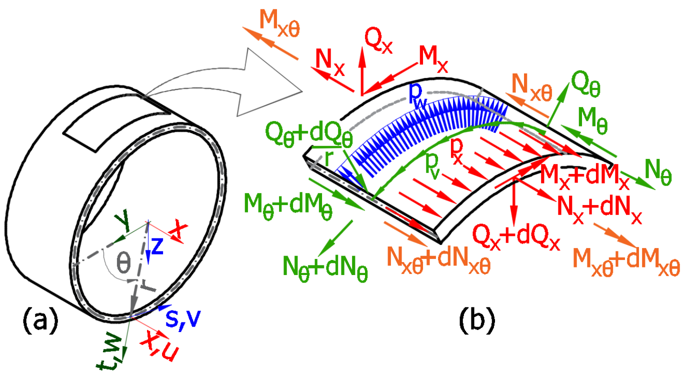

Figure 1: a) Coordinate systems adopted in GBT's analysis of thin-walled circular hollow section; b) orientation of the internal and external forces.

is expressed by:

$$u(x,\theta)=\sum_{i=1}^{n}{}^{i}u(\theta)\,{}^{i}V'(x) \tag{1}$$

$$v(x,\theta)=\sum_{i=1}^{n}{}^{i}v(\theta)\,{}^{i}V(x) \tag{2}$$

$$w(x,\theta)=\sum_{i=1}^{n}{}^{i}w(\theta)\,{}^{i}V(x) \tag{3}$$

Here, the upper-left index i indicates GBT's deformation mode. It is important to note that the first derivative, $V'(x)$, in eq. 1 is not arbitrary, but it is necessary to link the longitudinal displacement in the cross-section, $u(x,\theta)$, with the transversal displacements: $v(x,\theta)$ and $w(x,\theta)$. Thus, one has a clear understanding of the functionality of each cross-section functions: i) $u(\theta)$ leads to warping; ii) $v(\theta)$ and $w(\theta)$ lead to cross-section distortion.

2.2 Hollow circular cross-section's analysis

One of the major characteristics of GBT is the cross-section analysis, which is only based on geometry and mechanical properties of the cross-section. Such analysis leads to n natural numbers of orthogonal mode shapes of deformation, where (i) the lower modes, i.e $i=1,2$ and 3, represent the longitudinal elongation, the major and minor bending directions; (ii) the higher modes $i>3$ involve cross-section distortions and ovalizations. Furthermore, in hollow circular cross-sections, two additional modes are introduced (Silvestre 2007): the pure axial extension mode $i=a$ and the pure Saint-Venant torsion mode $i=t$.

For a non-circular cross-section, the analysis starts in the assembling of stiffness matrices related to longitudinal, transversal and shear strains due to membrane and plate behavior of the structural profile. This assembling step has a non-trivial setup (Jönsson & Andreassen 2011, Jönsson & Andreassen 2012b, Jönsson & Andreassen 2012a, Bebiano 2015), especially for non-circular cross-sections, which leads to a generic eigenvalue problem for the lower modes and a quadratic eigenvalue problem for high modes (Jönsson & Andreassen 2012b). Fortunately, in the case of circular hollow sections this laborious step is replaced by orthogonal deformation shapes based on Fourier series (Schardt 1989) (Silvestre 2007). In figure 2 some of these deformation shapes are presented, and their respective values are given by:

- For pure axial extension mode, i = a:

$${}^{i}u(\theta)=0 \quad {}^{i}v(\theta)=0 \quad {}^{i}w(\theta)=1 \tag{4}$$

- For pure torsion mode, i = t:

$${}^{i}u(\theta)=0 \quad {}^{i}v(\theta)=1 \quad {}^{i}w(\theta)=0 \tag{5}$$

- For pure longitudinal extension mode, i = 1:

$${}^{i}u(\theta)=1 \quad {}^{i}v(\theta)=0 \quad {}^{i}w(\theta)=0 \tag{6}$$

- For odd modes, i = 3,5,7,... with m = (i-1)/2

$$\begin{cases}{}^{i}u(\theta)=-r\cos(m\theta)\\ {}^{i}v(\theta)=-m\sin(m\theta)\\ {}^{i}w(\theta)=m^{2}\cos(m\theta)\end{cases} \tag{7}$$

- For even modes, i = 2,4,6,... with m = i/2

$$\begin{cases}{}^{i}u(\theta)=r\sin(m\theta)\\ {}^{i}v(\theta)=-m\cos(m\theta)\\ {}^{i}w(\theta)=-m^{2}\sin(m\theta)\end{cases} \tag{8}$$

where r is the middle-line radius. From these functions for the orthogonal modes, one obtains the generalized cross-section properties based on each modal strain. For instance, the longitudinal strain given by:

$$\epsilon_{x}(x,\theta,t)=[u(\theta)-tw(\theta)]\,V''(x) \tag{9}$$

leads to the generalized warping inertia ${}^{i}C$, where t is the thickness of the cross-section's wall. Hence, the membrane ${}^{i}C^{M}$ and the plate ${}^{i}C^{P}$ stiffness are:

$${}^{i}C={}^{i}C^{M}+{}^{i}C^{P}=\oint\left(tr\,{}^{i}u(\theta)^{2}+\frac{Kr\,{}^{i}w(\theta)^{2}}{E}\right)\mathrm{d}\theta \tag{10}$$

with: E is Young's modulus; K is the plate stiffness given by $K=\frac{Et^{3}}{12(1-\mu^{2})}$ with μ as Poisson's ratio.

The transversal elongation strain is given by:

$$\epsilon_{\theta}(x,\theta,t)=\frac{\dot{v}(x,\theta)+w(x,\theta)}{r}-t\frac{w(x,\theta)+\ddot{w}(x,\theta)}{r^{2}} \tag{11}$$

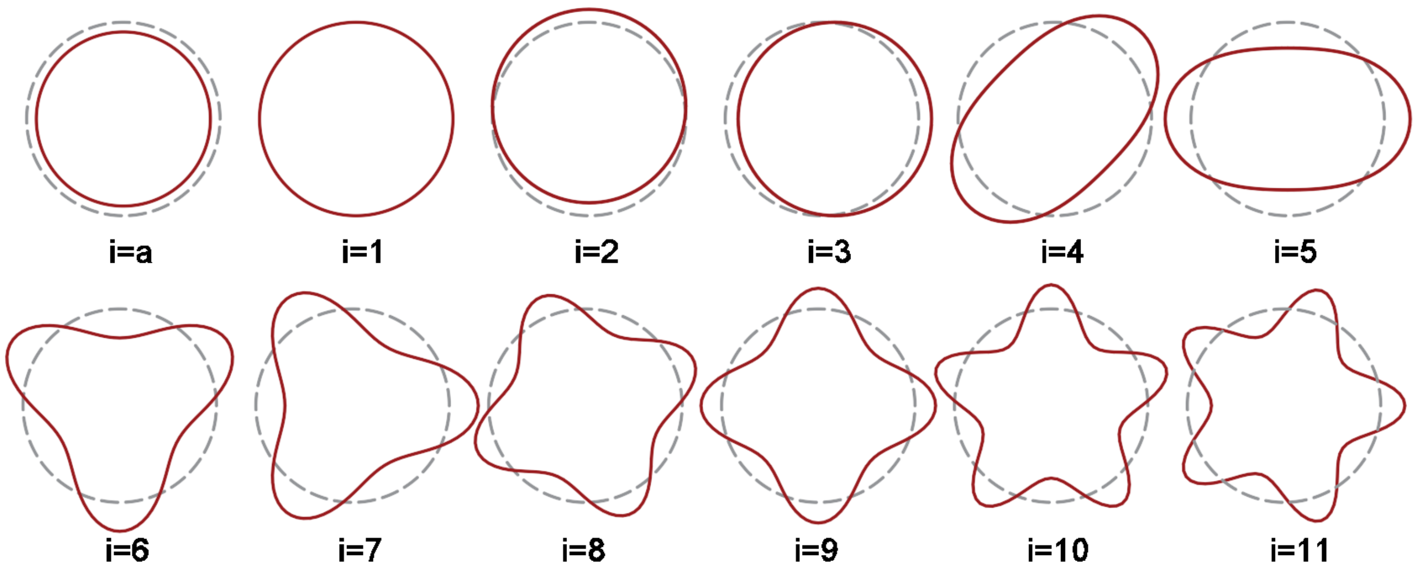

Figure 2: Transverse deformation shape modes of a thin-walled circular hollow section according to GBT.

where the dot index represents the derivative $d/\mathrm{d}\theta$. Since the first term in the above expression represents the membrane strain of transversal elongation, it is vanished according to traditional GBT approach. Therefore, one reaches the following restrain condition:

$$w(\theta) = -\dot{v}(\theta) \tag{12}$$

And the second term in eq. 11 leads to the generalized distortion stiffness:

$$^{i}B = \frac{K}{r^3} \oint \left({}^{i}w(\theta) + {}^{i}\ddot{w}(\theta) \right)^2 \mathrm{d}\theta \tag{13}$$

Concerning the shear strain, the first term in its expression:

$$\gamma_{x\theta}(x,\theta,t) = \frac{\dot{u}(x,\theta) + rv'(x,\theta)}{r} + \\ - t\frac{\dot{u}(x,\theta) - rv'(x,\theta) + 2r\dot{w}'(x,\theta)}{r^2} \tag{14}$$

is neglected as well in traditional GBT analysis. Therefore, one reaches the following restrain:

$$v(\theta) = -\frac{\dot{u}(\theta)}{r} \tag{15}$$

The second term in eq. 14 leads to the inertia concerning the cross-sectional shear and Poisson effect stiffness (${}^{i}D$ and ${}^{i}D_{\mu}$, respectively are given by

$$^{i}D = \frac{r}{3}t^3 \oint \left(\frac{{}^{i}\dot{w}(\theta) - {}^{i}v(\theta)}{r} \right)^2 \mathrm{d}\theta \tag{16}$$

$$^{i}D_{\mu} = r \oint \frac{{}^{i}w(\theta) + {}^{i}\ddot{w}(\theta)}{r^2} w(\theta)\,\mathrm{d}\theta \tag{17}$$

Evaluating the closed line integral in equations 10, 13, 16 and 17, one obtains the practical formulation:

$$^{i}C = \begin{cases} 0 \text{ for } i = t \\ 2\pi rK/E \text{ for } i = a \\ 2\pi rt \text{ for } i = 1 \\ \pi t r^3 \left(1 + \frac{t^2 m^4}{12r^2(1-\mu^2)}\right) \text{ for } i > 1 \end{cases} \tag{18}$$

$$^{i}D = \begin{cases} \pi \frac{t^3}{3r} m^2 (m^2 - 1) \left(\frac{m^2}{1-\mu} - 1\right) \text{ for } i > 1 \\ 0 \quad \text{for all other cases} \end{cases} \tag{19}$$

$$^{i}D_{\mu} = \begin{cases} \frac{\pi}{r} m^4 (1 - m^2) \text{ for } i > 1 \\ 0 \quad \text{for all other cases} \end{cases} \tag{20}$$

$$^{i}B = \begin{cases} 0 \text{ for } i = t \text{ and } i = 1 \\ 2\pi Et/r \text{ for } i = a \\ \pi \frac{K}{r^3} m^4 (m^2 - 1)^2 \text{ for } i > 1 \end{cases} \tag{21}$$

2.3 GBT's external force decomposition and orthogonal ordinary differential equation

From all infinite possible modes of GBT, only a few modes are required to evaluate a problem. Since GBT presents the external deformation energy also as modal superposition, the modal decomposition of the external loads filters the relevant modes. Likewise the internal strain energy, the external general loads functions $p_x(x,\theta)$, $p_v(x,\theta)$ and $p_w(x,\theta)$ are assumed to be represented by separation of variables:

$$p_x(x,\theta) = f_x(x)\, q_x(\theta) \tag{22}$$ $$p_v(x,\theta) = f_v(x)\, q_v(\theta) \tag{23}$$

$$p_w(x,\theta) = f_w(x)\, q_w(\theta) \tag{24}$$

One obtains the modal decomposition by the inner product of the deformation modes, given in equations 7 and 8, and the functions $q_x(\theta)$, $q_v(\theta)$ and

$q_w(\theta)$ of the external load:

$$^{i}q_x = -r \oint q_x(\theta)\, ^{i}u(\theta)\, \mathrm{d}\theta \quad (25)$$

$$^{i}q_v = r \oint q_v(\theta) ^{i}v(\theta) \mathrm{d}\theta \quad (26) \quad ^{i}q_w = r \oint q_w(\theta) ^{i}w(\theta) \mathrm{d}\theta \quad (27)$$

Thus, from the Hamilton's principle, one can reach the GBT's ordinary differential equation:

$$E^{i}C^{i}V''''(x) - (G^{i}D - 2\mu K^{i}D_{\mu})^{i}V''(x) + {}^{i}B^{i}V(x)$$

$$= f_x(x)\, ^{i}q_x(x) + f_v(x)\, ^{i}q_v(x) + f_w(x)\, ^{i}q_w(x) \quad (28)$$

And the modal amplification function $^{i}V(x)$ is obtained, which acts as a generalized beam problem.

3 COMPLEMENTARY TRANSVERSAL AND SHEAR MEMBRANE ENERGIES IN HOLLOW CIRCULAR CROSS-SECTION

This section presents an alternative approach in the evaluation of membrane's shear and transversal energies based on relaxation of the restrains conditions given in equations 12 and 15. Consequently, new set of transversal displacement modes are introduced in the formulation, as follow in the next sub-section.

3.1 Complementary transversal displacement functions

The new assumptions, in this alternative approach, are: i) an introduction of a complementary displacement function, which leads to any shear and transversal stress description; ii) these complementary functions must have a minimal perturbation in all usual GBT's formulation. Consequently, this second assumption requires, initially, the null Poisson's ration to avoid the coupling between transversal and longitudinal strain-stress of membrane kinematics. Nevertheless, once one develops the complementary modes, the coupling with the traditional GBT's mode can be addressed in the total energy formulation.

Following the first assumption, the natural choice to describe any additional transversal displacement in circular cross-section is again periodic functions. Hence, similar to the traditional GBT's modes, the complementary modes are defined as Fourier terms. Moreover, one can observe that among the three displacement functions: $u(\theta)$, $w(\theta)$ and $v(\theta)$ the first one is not required. In fact, GBT's longitudinal membrane strain, given in eq. 9, needs no additional term, which dismiss any extra term in $u(\theta)$. Thereby, the additional transversal modes have the following definition:

$$u_+(\theta) = 0 \quad (29)$$

$$v^{\star}_{+}(\theta) = \begin{cases} \frac{-sin(m\theta)}{m} \text{for odd mode} \\ \frac{cos(m\theta)}{m} \text{for even mode} \end{cases} \quad (30)$$

$$w_+(\theta) = \begin{cases} \frac{cos(m\theta)}{m^2} \text{for odd mode} \\ \frac{sin(m\theta)}{m^2} \text{for even mode} \end{cases} \quad (31)$$

The sub-index $_+$ is used here to differentiate the additional function in respect to the usual displacement function. Since, equations. 30 and . 31 have similar definition of the traditional GBT's modes, the only possible way to keep the second assumption above is the by adopting an amplification function in x different of $V(x)$, named here as $S(x)$. Hence, the fully complementary transversal displacement function is:

$$v_+(x,\theta) = \sum_{i=1}^{n} {}^{i}v(\theta)\, ^{i}S(x); \quad (32)$$

$$w_+(x,\theta) = \sum_{i=1}^{n} {}^{i}w(\theta)\, ^{i}S(x) \quad (33)$$

3.2 Complementary transversal and shear strains functions

The introduction of equations 29, 30 and 31 into the expressions of the transversal and shear strains (equations 11 and 14 respectively) leads to a straightforward evaluation:

$$\epsilon_{+\theta}(x,\theta,t) = \frac{\dot{v}_+(x,\theta) + w_+(x,\theta)}{r}$$

$$- t\frac{w_+(x,\theta) + \ddot{w}_+(x,\theta)}{r^2} \quad (34)$$

$$\gamma_{+x\theta}(x,\theta,t) = v'_+(x,\theta) - t\frac{v'_+(x,\theta) - 2\dot{w}'_+(x,\theta)}{r} \quad (35)$$

3.3 Complementary internal strain energy

Once the complementary strains are defined, one can develop the internal strain energy:

$$U_{int+} = \int_V \left[\int_{\epsilon^M_{+\theta}} \sigma^M_{+\theta} \mathrm{d}\epsilon^M_{+\theta} + \int_{\gamma^M_{+x\theta}} \tau^M_{+x\theta} \mathrm{d}\gamma^M_{+x\theta} \right] \mathrm{d}V \quad (36)$$

Which the variation is evaluated by the introduction of expression 32- 35:

$$\delta U_{int+} = \sum_{i=1}^{n} \int_V G\left(\gamma_{+x\theta}(x,\theta,t)\right)^{2i} S'(x)\, \delta^{i} S'(x)$$

$$+ E\left(\epsilon_{+\theta}(x,\theta,t)\right)^{2}\, ^{i}S(x)\, \delta\, ^{i}S(x)\, \mathrm{d}V \quad (37)$$

The integrals over the area of the quadratic terms above, with the introduction of equations 34 and

35 lead to the complementary cross-section properties:

$$^{i}A_s = r\oint \left(\gamma_{+x\theta}\left(x,\theta,t\right)\right)^2 \mathrm{d}\theta = \frac{\pi r t}{m^2}\left(1+\frac{t^2}{12r^2}\right) \quad (38)$$

$$^{i}A_\theta = r\oint \left(\epsilon_{+\theta}\left(x,\theta,t\right)\right)^2 \mathrm{d}\theta = \frac{\pi t}{r}\left(\frac{1-m^2}{m^2}\right)^2\left(1+\frac{t^2}{12r^2}\right) \quad (39)$$

Here, one can recognize that, if the transversal deformation modes related to Bernoulli-Euler beam are choosen ($m=1$), then A_s is nothing less that the product of Timoshenko shear factor and the cross-section area.

3.4 Complementary external load potential energy

Following the same approach presented in section 2.3, the external load functions are decomposed according to the principals of separation of variables:

$p_v\left(x,\theta\right) = f_v\left(x\right) q_{v_+}$ (40) $p_w(x,\theta) = f_w\left(x\right) q_{w_+}$ (41)
where q_{v_+} and q_{w_+} can be expressed as modal superposition from the decomposition of external loads, given by the inner products:

$$^{i}q_{v_+} = r\oint q_v\left(\theta\right)\, ^{i}v_+\left(\theta\right)\mathrm{d}\theta \quad (42)$$

$$^{i}q_{w_+} = r\oint q_w\left(\theta\right)\, ^{i}w_+\left(\theta\right)\mathrm{d}\theta \quad (43)$$

3.5 Equilibrium by Hamilton's principle in Transversal and Shear Membrane Energy

In order to obtain the ordinary differential equation of equilibrium in the complementary transversal and shear modes, the Hamilton's principle receives the internal and external energy variations:

$$\int_L G^i A_s{}^i S'\left(x\right)\delta^i S'\left(x\right) + E^i A_\theta{}^i S\left(x\right)\delta^i S\left(x\right) - {}^{i}q_{v_+} f_v\left(x\right)\delta^i S\left(x\right) - {}^{i}q_{w_+} f_w\left(x\right)\delta^i S\left(x\right)\mathrm{d}x = 0 \quad (44)$$

All terms involving the first derivative of variated function, $\delta^i S'$ are integrated by parts, which the final results, for each mode, leads to:

$$\int_L \left[-G^i A_s{}^i S''\left(x\right) + E^i A_\theta{}^i S\left(x\right) + {}^{i}q_{v_+} f_v\left(x\right) - {}^{i}q_{w_+} f_w(x)\right]\mathrm{d}x\delta^i S\left(x\right) + \left[\left(G^i A_s{}^i S'\left(x\right)\right)\delta^i S(x)\right]_{in}^{fi} \quad (45)$$

Here, the equilibrium and the boundary conditions stand out. Taking into account that the functional eq. 45 must be vanished for any arbitrary functions of longitudinal amplification, $\delta^i S\left(x\right)$, the parentheses terms of the integral must be zero, which gives the non-homogeneous differential equation of equilibrium in transversal and shear complementary energy in GBT:

$$-G^i A_s{}^i S''\left(x\right) + E^i A_\theta{}^i S\left(x\right) = -{}^{i}q_{v_+} f_v\left(x\right) + {}^{i}q_{w_+} f_w\left(x\right) \quad (46)$$

And the boundary conditions are found in the remained terms:

$$\left[\left(G^i A_s{}^i S'\left(x\right)\right)\delta^i S(x)\right]_{in} = 0 \quad (47)$$

$$\left[\left(G^i A_s{}^i S'\left(x\right)\right)\delta^i S(x)\right]_{fi} = 0 \quad (48)$$

4 NUMERICAL EXAMPLE

As a numerical example, a thin-walled circular hollow steel cross-section used as a segment of 1m of a buried pipeline is shown in figure 3. This cross-section is under a downward and upward linear projected surface load, which represents the earth pressure. I.e., the total load applied on the structure is not a product of the surface load and the area of the surface, but it is the product of the surface load and the projected area on the global coordinate direction z. The material parameters are Young's modulus $E = 205,000N/mm^2$, Poisson's ratio $\mu = 0.0$, and shear modulus $G = 78,846.2N/mm^2$.

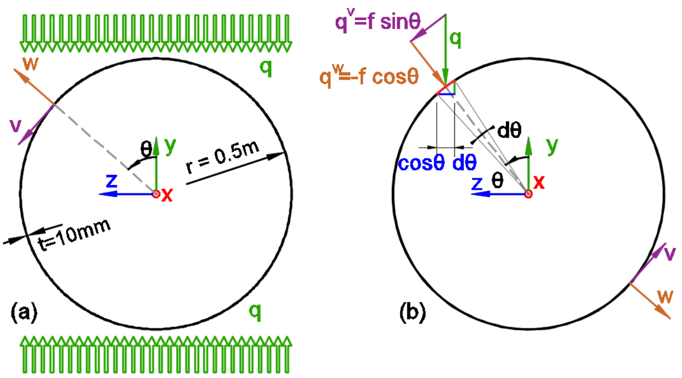

Figure 3: Buried pipeline segment under downward and upward pressure: a) layout configuration; b)pressure load in polar coordinates.

4.1 Setup of finite element and GBT Analysis

The projected pressure in figure 3.b leads to the following expressions: $q_v = q\cos\left(\theta\right)\sin\left(\theta\right)$ and $q_w = -q\cos^2\left(\theta\right)$, which has in the GBT modal decomposition only participation in mode a and 5. Hence, one obtains: ${}^{a}q_v = 0$, ${}^{a}q_w = -50\pi$, ${}^{5}q_v = -12.5\pi$

and ${}^5q_w = -6.25\pi$ (in N/mm). After the filtering of GBT modes, one needs to solve the GBT's differential equations, given in 46. Since this differential equation is the same type of classical Vlasov Beam Theory, the exact finite element solution presented in (Magbi 2003) is adopted and adapted with the replacement of correspondent inertias. Form this finite element analysis one obtains the following modal amplification functions :${}^aV(x) = -6.097E-3$ and ${}^5V(x) = -8.130E-3$ (in mm). As continuous interpolation functions, all derivatives are vanished. From this displacement field, one obtains the transversal membrane force:

$$N_\theta\left(x,\theta\right)=\frac{Et}{r}\left({}^aV\left(x\right)+\sum_{i=2}^{n}\left({}^i\dot{v}_+\left(\theta\right)+{}^iw_+\left(\theta\right)\right){}^iS\left(x\right)\right) \tag{49}$$

Which leads to modes a and 5 to:

$$N_\theta\left(x,\theta\right) = -25\left(1-\cos\left(2\theta\right)\right) \tag{50}$$

Below, the diagram of transversal membrane internal force and the result obtained in a control shell model elaborated from commercial ANSYS® software are presented and compared: The mean

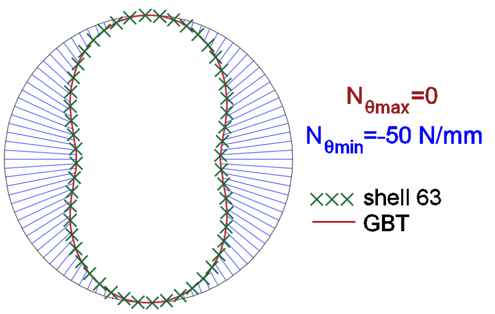

Figure 4: Comparison of results among mixed GBT and fully shell models: transversal membrane force, N_θ

difference between the models are around 0.2%.

5 CONCLUSIONS

This study presents an extension in GBT in order to include the shear and transversal elongation energies of membrane behavior. Based on additional mode shapes, which are nothing less than a complementary term in transversal displacement functions $v(\theta)$ and $w(\theta)$, the restriction of null neither shear nor transversal elongation energies are relaxed. For bending moment, this additional displacement function leads to the shear stiffness of Timoshenko Beam Theory. Meanwhile, for the higher modes the complementary displacement functions add a minute difference in the displacement field, but with a directed assessment of the shear and transversal membrane force. The numerical example shows an almost exact agreement between the a fully shell element model and the proposed GBT model.

6 ACKNOWLEDGE

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